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Journal of Combinatorial Theory, Series A 108 (2004) 199–204
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

# On a hypercube coloring problem

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Received 10 March 2004

Communicated by Günter Ziegler
Available online 18 August 2004

**Abstract**

Let $\chi_{\bar{k}}(n)$ denote the minimum number of colors necessary to color the $n$-dimensional hypercube so that no two vertices that are at a distance at most $k$ from each other get the same color. In other words, this is the smallest number of binary codes with minimum distance $k+1$ that form a partition of the $n$-dimensional binary Hamming space. It is shown that $\chi_{\bar{2}}(n) \sim n$ and $\chi_{\bar{3}}(n) \sim 2n$ as $n$ tends to infinity.


*Keywords:* Binary code; Chromatic number; Hypercube; Vertex coloring

## 1. Introduction

The $n$-dimensional hypercube, or $n$-cube, can be formed by taking one vertex for each binary $n$-tuple, two vertices being adjacent exactly when the Hamming distance between the corresponding $n$-tuples is 1. Here, we study the following vertex-coloring problem for $n$-cubes: What is the value of $\chi_{\bar{k}}(n)$, the minimum number of colors necessary to color the vertices of the $n$-cube so that no two vertices that are at distance at most $k$ from each other get the same color? In other words: What is the chromatic number of the $k$th power of the $n$-cube?

In coding-theoretical terms, $\chi_{\bar{k}}(n)$ is the smallest number of binary codes with minimum distance $k+1$ that form a partition of the $n$-dimensional binary Hamming space, $Q^n$.

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[1] Supported by the Academy of Finland under Grants No: 100500 and 202315.

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doi:10.1016/j.jcta.2004.06.010

Several recent papers [5,6,8,10,11] have addressed the problem of determining (bounds on) $\chi_{\bar{k}}(n)$ for general $k$ and, in particular, for the smallest nontrivial cases $k = 2, 3$; see also [3, Section 9.7]. These studies have also addressed the related problem of determining $\chi_k(n)$, the minimum number of colors when no vertices at a distance *exactly* $k$ from each other are allowed to have the same color.

It is known that

$$n + 1 \le \chi_{\bar{2}}(n) \le 2^{\lceil \log_2(n+1) \rceil} \tag{1}$$

and

$$2n \le \chi_{\bar{3}}(n) \le 2^{\lceil \log_2 n \rceil + 1}, \tag{2}$$

see [10] and [5], respectively.

Since the color classes of a coloring attaining $\chi_{\bar{k}}(n)$ can be viewed as binary codes with minimum distance $k + 1$,

$$\chi_{\bar{k}}(n) \ge \frac{2^n}{A(n, k+1)}, \tag{3}$$

where $A(n, d)$ is the maximum size of a binary code with length $n$ and minimum distance $d$. Binary Hamming codes attain the value $A(2^t - 1, 3) = 2^{2^t - t - 1}$. Best and Brouwer [1] proved the stronger result that $A(2^t - i - 1, 3) = 2^{2^t - t - i - 1}$ for $0 \le i \le 3$, which together with (3) implies that $\chi_{\bar{2}}(2^t - i - 1) \ge 2^t$ for $0 \le i \le 3$, and (1) then gives that

$$\chi_{\bar{2}}(2^t - i - 1) = 2^t \quad \text{for } 0 \le i \le 3.$$

This result made some people conjecture that $\chi_{\bar{2}}(n)$ is always a power of two [10], but this conjecture is disproved by $13 \le \chi_{\bar{2}}(8) \le 14$, obtained independently by Hougardy in 1991 [11] and Royle in 1993 [3, Section 9.7].

By (1) and (2), a factor of 2 separates the best known asymptotic upper and lower bounds on $\chi_{\bar{2}}(n)/n$ and $\chi_{\bar{3}}(n)/n$, and similarly for $\chi_2(n)/n$ (see Lemma 2). In fact, it turns out that the limits of these fractions exist: in the current paper it is proved that the lower bounds rather than the upper bounds in (1) and (2) have the same asymptotic behavior as the respective functions. The proof technique is the same as in the proof of an analogous result for the dual problem of determining $R$-domatic numbers of $n$-cubes [9], which in turn directly utilizes ideas by Kabatyanskii and Panchenko [4]. Throughout the paper, a coloring is viewed as a partition of $Q^n$ into codes (color classes).

## 2. The result

The main theorem to be proved concerns $\chi_{\bar{2}}(n)$. By the following lemmata, analogous results can then be obtained for $\chi_{\bar{3}}(n)$ and $\chi_2(n)$.

**Lemma 1.** $\chi_{\bar{3}}(n + 1) \le 2\chi_{\bar{2}}(n)$.

**Proof.** From each code of a coloring attaining $\chi_{\bar{2}}(n)$, construct two new codes by extending the code with a parity-check bit for odd and even parity, respectively. □

The inverse process to extending a code is called puncturing, and consists of deleting one or more coordinates from each codeword.

**Lemma 2.** $\chi_2(n) = \chi_{\bar{2}}(n-1)$.

**Proof.** (see also Ziegler [11]). If two words are at a distance 2 from each other, then their weights have the same parity. Hence, the even- and the odd-weight words of a coloring attaining $\chi_2(n)$ can be considered separately. Without loss of generality, consider the even-weight words. Two words in the same code (color class) must be at a distance at least 4 from each other. By puncturing in any coordinate, we get $\chi_{\bar{2}}(n-1) \leq \chi_2(n)$.

Take a coloring attaining $\chi_{\bar{2}}(n-1)$ and lengthen the codes by forming the code $\{|\, x \,|\, 0 \,| : x \in C\} \cup \{|\, x \,|\, 1 \,| : x \in C\}$ from a code $C$. The distance between two words in a code is then 1 or $\geq 3$, so $\chi_2(n) \leq \chi_{\bar{2}}(n-1)$. □

For completeness, we reprove the following lower bounds [5,10], which we have already seen in (1) and (2).

**Lemma 3.** $\chi_{\bar{2}}(n) \geq n+1$, $\chi_{\bar{3}}(n) \geq 2n$.

**Proof.** The size of a clique gives a lower bound on the chromatic number of a graph. Consequently, a lower bound on $\chi_{\bar{k}}(n)$ is obtained from the size of a clique in the $k$th power of the $n$-cube. For $k = 2$, such a clique is formed by the all-zero word and the $n$ words of weight 1. For $k = 3$, a clique is formed by the words with weight 0 or 1 in the first coordinate and weight 0 or 1 in the last $n-1$ coordinates. □

To prove the main theorem we need three bounds, adopting the approach in [4]. The first bound is easy to see.

**Lemma 4.** $\chi_{\bar{2}}(n-1) \leq \chi_{\bar{2}}(n)$.

**Proof.** Given a coloring attaining $\chi_{\bar{2}}(n)$, consider the coloring induced by the words with a 0 in the last coordinate, and remove that coordinate. □

The $|\, u \,|\, u+v \,|$ construction [7, Chapter 2] is an effective construction of new codes from old ones and is the core of the next theorem.

**Theorem 1.** $\chi_{\bar{2}}(2n+1) \leq 2\chi_{\bar{2}}(n)$.

**Proof.** For each code $C$ in a coloring corresponding to $\chi_{\bar{2}}(n)$, we construct two codes consisting of the words $|\, u \,|\, u+v \,|\, \pi(u) \,|$ and $|\, u \,|\, u+v \,|\, \overline{\pi(u)} \,|$ for $u \in Q^n$ and $v \in C$, where $\pi(u) = \mathrm{wt}(u) \bmod 2$ and $\overline{\pi(u)} = (\mathrm{wt}(u)+1) \bmod 2$. The new codes all have minimum distance at least 3 and are disjoint. The result now follows, since the total number of words in all new codes is $2 \times 2^n \times 2^n = 2^{2n+1}$ and there are $2\chi_{\bar{2}}(n)$ codes. $\square$

In the following, we need nonbinary codes. For the coordinate values of a $q$-ary code, we use the set $\{0, 1, \ldots, q-1\}$.

**Theorem 2.** *Let $p > 2^r$ be a prime power and let $i \geq 2$. Then*

$$\chi_{\bar{2}}\left(\frac{(p^i-1)(2^r-1)}{p-1}\right) \leq p^i.$$

**Proof.** Since $p$ is a prime power, there is a $p$-ary Hamming code of length $(p^i-1)/(p-1)$ for any $i \geq 2$. Such a Hamming code and its cosets—a total of $p^i$ codes—have minimum distance 3 and partition the $(p^i-1)/(p-1)$-dimensional $p$-ary Hamming space. From these codes an equal number of $2^r$-ary codes (with the same length) are constructed by deleting all codewords that have coordinate values outside the set $\{0, 1, \ldots, 2^r-1\}$. This way, we obtain a coloring of the $(p^i-1)/(p-1)$-dimensional $2^r$-ary Hamming space.

Let $C_i$, $0 \leq i \leq 2^r-1$ denote the cosets of the binary Hamming code of length $2^r-1$. The $2^r$-ary codes of the coloring obtained in the first part of the proof are now further transformed by replacing any coordinate value $a$ by codewords from $C_a$ in all possible ways:

$$(a_1, a_2, \ldots) \to \{|\, c_1 \,|\, c_2 \,|\, \cdots \,|\, : c_i \in C_{a_i}\}.$$

Hence, the codes obtained—there are still $p^i$ codes (color classes)—partition the $(p^i-1)(2^r-1)/(p-1)$-dimensional binary Hamming space and form a desired coloring. $\square$

We also need the following result on the distribution of primes; see, for example, [2]. Let $p(x)$ denote the smallest prime greater than $x$.

**Lemma 5.** $\lim_{x\to\infty} p(x)/x = 1$.

We are now ready for proving the main theorem of this paper by combining Lemma 4, Theorem 1 and Theorem 2. Once again we stress that this proof very closely follows those in [4,9].

**Theorem 3.** $\lim_{n\to\infty} \chi_{\bar{2}}(n)/n = 1$.

**Proof.** For fixed $p$ and $r$, $p > 2^r$, applying Theorem 1 to the bound in Theorem 2 gives the following result:

$$\chi_{\bar{2}}\left(\frac{2^j(p^i-1)(2^r-1)}{p-1} + (2^j-1)\right) \leq 2^j p^i.$$

For lengths

$$l(i,j) = \frac{2^j(p^i-1)(2^r-1)}{p-1} + (2^j-1),$$

we get

$$\begin{aligned}\frac{\chi_{\bar{2}}(l(i,j))}{l(i,j)} &\le \frac{2^j p^i(p-1)}{2^j(p^i-1)(2^r-1)+(p-1)(2^j-1)}\\ &\le \frac{p^i(p-1)}{(p^i-1)(2^r-1)}\\ &= \frac{(p^i-1)(p-1)+(p-1)}{(p^i-1)(2^r-1)} \le \frac{(p-1)+1}{(2^r-1)}\\ &= \frac{p}{2^r-1}. \end{aligned} \tag{4}$$

Recall that $\chi_{\bar{2}}(n) \ge n+1$ by Lemma 3, so $\chi_{\bar{2}}(n)/n$ is bounded from below by $(n+1)/n = 1+1/n$.

For better understanding of the rest of the proof, one should think of $l(i, j)$ in the framework of sequences, where a certain sequence is obtained by fixing $i$ and the elements of the sequence are obtained by varying $j$. To get values (lengths) that are not in any sequence $l(i, j)$, we apply Lemma 4. Application of Lemma 4 is the weakest part of the construction; however, we shall see that the sequences are sufficiently close to each other to get the desired value of the limit. It turns out that the particular length $l(i, r)$ is close to $l(i+1, 0)$:

$$\begin{aligned}\frac{l(i+1,0)}{l(i,r)} &= \frac{p^{i+1}-1}{2^r(p^i-1)+p-1}\\ &= \frac{p(p^i-1)+(p-1)}{2^r(p^i-1)+(p-1)}\\ &\le \frac{p}{2^r}. \end{aligned} \tag{5}$$

From (5) and $l(i, j+1) = 2l(i, j)+1$ we get that $l(i+1,k)/l(i,k+r) \le p/2^r$ for any $k \ge 0$. Combining this result with Lemma 4 and (4) gives that for lengths $l(i,k+r) \le n < l(i+1,k)$,

$$\frac{\chi_{\bar{2}}(n)}{n} \le \frac{p^2}{2^r(2^r-1)},$$

which by Lemma 5 can be made arbitrarily close to 1 when $n$ tends to infinity.

It is straightforward to show that $l(i+1,k)/l(i,k+r) > 1$ and that there is a function $f(p,r)$ such that

$$\frac{l(i+1,k)}{l(i,k+r)} - 1 > f(p,r) > 0. \tag{6}$$

It still remains to show that there is an $n'$ such that all lengths $n \ge n'$ belong to some interval $l(i,k+r) \le n < l(i+1,k)$. To prove this we use (6), which implies that there is an $m$ such that $(1+f(p,r))^m > 2$; we consider the smallest such $m$. Now $l(i+m,k) > 2l(i,k+mr)$ and, since $l(i,k+mr+1) = 2l(i,k+mr)+1$, we get that $l(i+m,k) \ge l(i,k+mr+1)$.

Ignoring the precise value of $j$ needed for synchronization and talking about these sequences on a more general level, this means that the elements of the sequence $l(i+1, j)$

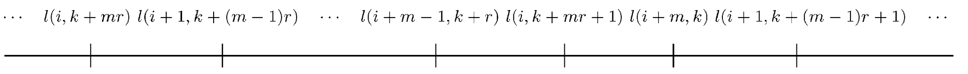


Fig. 1. Sequences of lengths.

are slightly greater than the elements of the sequence $l(i, j)$, the elements of $l(i+2, j)$ are slightly greater than those of $l(i+1, j)$, and so on, until the elements of $l(i+m-1, j)$, which are slightly smaller than those of $l(i, j)$, and the elements of $l(i+m, j)$, which are slightly greater than those of $l(i, j)$. See Fig. 1, where the precise values of the second parameter are shown. We may then choose $i = 2$ and $k = 0$ to get $n' = l(2, mr)$. □

By Theorem 3 and Lemmata 1–3, the asymptotic behavior of $\chi_{\bar{3}}(n)$ and $\chi_2(n)$ follows. Corollary 2 answers a question in [6].

**Corollary 1.** $\lim_{n\to\infty} \chi_{\bar{3}}(n)/n = 2$.

**Corollary 2.** $\lim_{n\to\infty} \chi_2(n)/n = 1$.

We conclude the paper by noticing that it is possible to generalize these results for colorings of $q$-ary Hamming spaces.

## Acknowledgements

The author thanks Günter Ziegler and the referees for helpful comments and suggestions.

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